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Conservative integrals of adiabatic Durran's equations

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SUMMARY

Potential advances are investigated in the area of generalized anelastic approximations. Consistent control-volume integrals are designed and compared for the established Lipps–Hemler form (of anelastic approximation) and Durran's pseudo-incompressible form. The Durran system provides a unique theoretical tool—useful for research of geophysical and stellar flows—within the existing set of reduced, Boussinesq-type fluid models. It represents thermal aspects of compressibility free of sound waves, yet the momentum equation is unapproximated. The latter admits unabbreviated baroclinic production of vorticity, thus facilitating separation of compressibility and baroclinicity effects *per se*. Compared with other reduced fluid models, there is little cumulative experience with integrating the Durran system. Perhaps the first conservative integrations of Durran's equations are presented, using flux-form transport methods and exact projection for the associated elliptic problem. Because the resulting code is built from a preexisting anelastic model, the consistency of the numerics is assured thus minimizing uncertainties associated with *ad hoc* code comparisons. While broader physical implications are addressed, theoretical considerations are illustrated with examples of atmospheric flows. Copyright © 2007 John Wiley & Sons, Ltd.

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1. INTRODUCTION

Boussinesq approximations[‡] underlie the majority of research in low Mach number flows under gravity, such as atmospheres (planetary and stellar) and oceans. Yet in such models the 3D baroclinic production of vorticity is abbreviated, in essence, to horizontal gradients of buoyancy—thus

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‡This includes anelastic equations that extend classical incompressible Boussinesq model [1] to a continuously stratified reference density [2].

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admitting thermally driven circulations only in vertical planes,[§] e.g. sea-breeze circulation.[¶] This fundamental departure from the full 3D baroclinicity of the complete Euler/Navier–Stokes' equations limits degrees of freedom available for realizations of Boussinesq fluids. The mathematical formalism underlying derivations of anelastic systems is a judicious scale analysis [1, 2, 4, 5] leading to reduced fluid model with fundamental mass, energy and entropy invariants, essential for the physical realizability of solutions. The validity of anelastic models cannot be assured *a priori*,[‖] and it depends on the portfolio of documented applications. Notwithstanding the success of anelastic models in advancing the understanding of geophysical and stellar flows, there are classes of fluid motions seemingly susceptible to the underlying assumptions yet sensitive to details of employed equations. One such example is the 'climate' of the solar convection zone (outer third of the Sun) whose simulations appear sensitive to the details of viscosity or baroclinicity; cf. [6, 7] and references therein.

Durran's [8] pseudo-incompressible approximation—also referred to as a subseismic approximation, in astrophysics [9]—is distinct. It retains the full form of the momentum equations, while filtering sound waves from the equations of motion, in the spirit of the anelastic models. While this accurately represents many aspects of planetary atmospheres [10], more important, it conveniently separates the baroclinicity from the compressibility effects, thus enabling inquiries into the role of baroclinicity in low Mach number flows. In particular, it may shed light on the dynamics of flows where compressible and anelastic models appear to predict contrasting realizations, such as in the solar convection zone [7] or bifurcating Rayleigh–Bénard convection of cryogenic gas [11]. Clearly, the distinct feature of the Durran pseudo-incompressible system makes it a unique theoretical tool that complements both the standard anelastic and fully compressible models.

In general, there is little experience with integrating even the simplified adiabatic form of Durran's equations; especially, in the strong conservation form with the exact projection for pressure. The mathematically consequential difference between the anelastic and the pseudo-incompressible system is the nonlinearity of the pressure-gradient force that ultimately leads to a nonlinear boundary value problem for pressure and alters the properties of the associated elliptic operator. In this paper, we extend the anelastic nonhydrostatic Eulerian/semi-Lagrangian numerical model for fluids EULAG—see [12, 13] for recent developments and summaries—to allow for integrating the adiabatic Durran equations, consistently with EULAG's proven semi-implicit nonoscillatory forward-in-time (NFT) numerics based on the MPDATA transport methods [14]. To retain EULAG's numerical structure for the Durran system, we design an iterative procedure, where at each iteration a linear elliptic problem—implied by the mass continuity constraint—is solved with a preconditioned nonsymmetric Krylov-subspace approach.

In the following section we describe the adiabatic Durran system and summarize the integration approach. Section 3 outlines numerical examples in the area of inertia-gravity wave dynamics, spawning the range of scales from meso to planetary as well as baroclinic life cycle experiments relevant to the formation of synoptic-scale weather systems. Remarks in Section 4 address briefly diabatic extensions and conclude the paper.

[§]The vertical is defined by the direction of gravity.

[¶]For example of some inadmissible effects, see discussion of Figure 10.8 in Section 10.4 in [3].

[‖]This is because the scale analysis is rigorously discriminating only when it indicates that some terms in the equations cannot be neglected; obviously, even small terms can have profound impact on the solutions for fluid states close to bifurcation.

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2. PSEUDO-INCOMPRESSIBLE MODEL

2.1. Analytic formulation

It is constructive to write an adiabatic Durran's system [8] in a symbolic perturbation form conforming with EULAG's default formulation [12, 13] built on the anelastic Lipps–Hemler equations [4]:

$$\nabla\cdot(\rho^*\mathbf{v})=0;\quad \frac{\mathrm{D}\theta'}{\mathrm{D}t}=-\mathbf{v}\cdot\nabla\theta_{\mathrm{e}};\quad \frac{\mathrm{D}\mathbf{v}}{\mathrm{D}t}=-\theta\nabla\pi'-\mathbf{g}\frac{\theta'}{\theta_{\mathrm{e}}}-\mathbf{f}\times\left(\mathbf{v}-\frac{\theta}{\theta_{\mathrm{e}}}\mathbf{v}_{\mathrm{e}}\right) \tag{1}$$

Here ρ^* denotes a generalized density, θ is the potential temperature, vectors $\mathbf{g}$ and $\mathbf{f}$ are the gravity acceleration and the Coriolis parameter, π' is a normalized pressure-perturbation variable; and primes symbolize deviations from geostrophically balanced environmental (ambient) state $(\mathbf{v}_{\mathrm{e}}, \theta_{\mathrm{e}})$, implied by the governing equations. The pseudo-incompressible system (1) and the Lipps–Hemler anelastic system differ in two aspects. First, $\rho^*=\rho_{\mathrm{b}}\theta_{\mathrm{b}}$ in (1) but $\rho^*=\rho_{\mathrm{b}}$ in the anelastic mass-continuity equation—subscript b refers to a static horizontally homogeneous reference state. Second, the momentum equation in (1) is unapproximated, whereupon factors $\propto\theta$ appear in the pressure-gradient and Coriolis accelerations, and θ_{e} replaces θ_{b} in the denominator of the buoyancy term. In effect, the deeper and broader the studied atmosphere, and/or larger the stratification, the greater will be solution departures from the familiar behaviors of anelastic codes.

2.2. Numerical approximations

The NFT algorithm employed in EULAG to integrate (1) (or its equivalent Eulerian flux form) can be formally written as

$$\psi_i^{n+1}=\mathrm{LE}_i(\tilde{\psi})+0.5\delta t R_i^{n+1}\equiv\widehat{\psi}_i+0.5\delta t R_i^{n+1} \tag{2}$$

where ψ_i^{n+1} is the solution sought at the grid point $(t^{n+1}, \mathbf{x}_i)$, $\tilde{\psi}\equiv\psi^n+0.5\delta t R^n$, and LE denotes a two-time level either advective semi-Lagrangian [15] or flux-form Eulerian [16] nonoscillatory two-time level transport operator (viz advection scheme). Equation (2) represents a system implicit with respect to all dependent variables in (1), because velocity, pressure and potential temperature are assumed to be unknown at $n+1$. In order to retain this numerical structure for the Durran system, algorithm (2) is executed within an outer iteration:

$$\begin{aligned}\theta'|_i^{n+1,v}&=\widehat{\theta'}_i-0.5\delta t(\mathbf{v}^{n+1,v}\cdot\nabla\theta_{\mathrm{e}})_i\\ \mathbf{v}_i^{n+1,v}&=\widehat{\mathbf{v}}_i-0.5\delta t\left[\theta^{n+1,v-1}\nabla\pi'|^{n+1,v}+\mathbf{g}\frac{\theta'|^{n+1,v}}{\theta_{\mathrm{e}}}+\mathbf{f}\times\left(\mathbf{v}^{n+1,v}-\frac{\theta^{n+1,v-1}}{\theta_{\mathrm{e}}}\mathbf{v}_{\mathrm{e}}\right)\right]_i\end{aligned} \tag{3}$$

where $v=1,\ldots,m$ numbers the outer iterations, and at each iteration the linear elliptic problem—implied by the first equation in (1)—is solved using a preconditioned generalized conjugate-residual (GCR) approach [17, 18]. Note that the only elements lagged behind in (3) are the $\propto\theta$ factors in the pressure-gradient and Coriolis accelerations. For the first guess $\theta^{n+1,0}$ we use the homogeneous solution (2) for $\psi\equiv\theta=\theta'+\theta_{\mathrm{e}}$ and $R\equiv 0$. Typically, the outer iteration converges rapidly, and the initial guess for $\mathbf{v}^{n+1,v}$ attains a suitable 'physically meaningful' convergence threshold of the GCR solver—$\|(\delta t/\rho^*)\nabla\cdot(\rho^*\mathbf{v})\|_\infty<\varepsilon$, cf. [19]—already at $v=m=3$. Correspondingly, the work within the GCR solver decreases rapidly past $v=1$.

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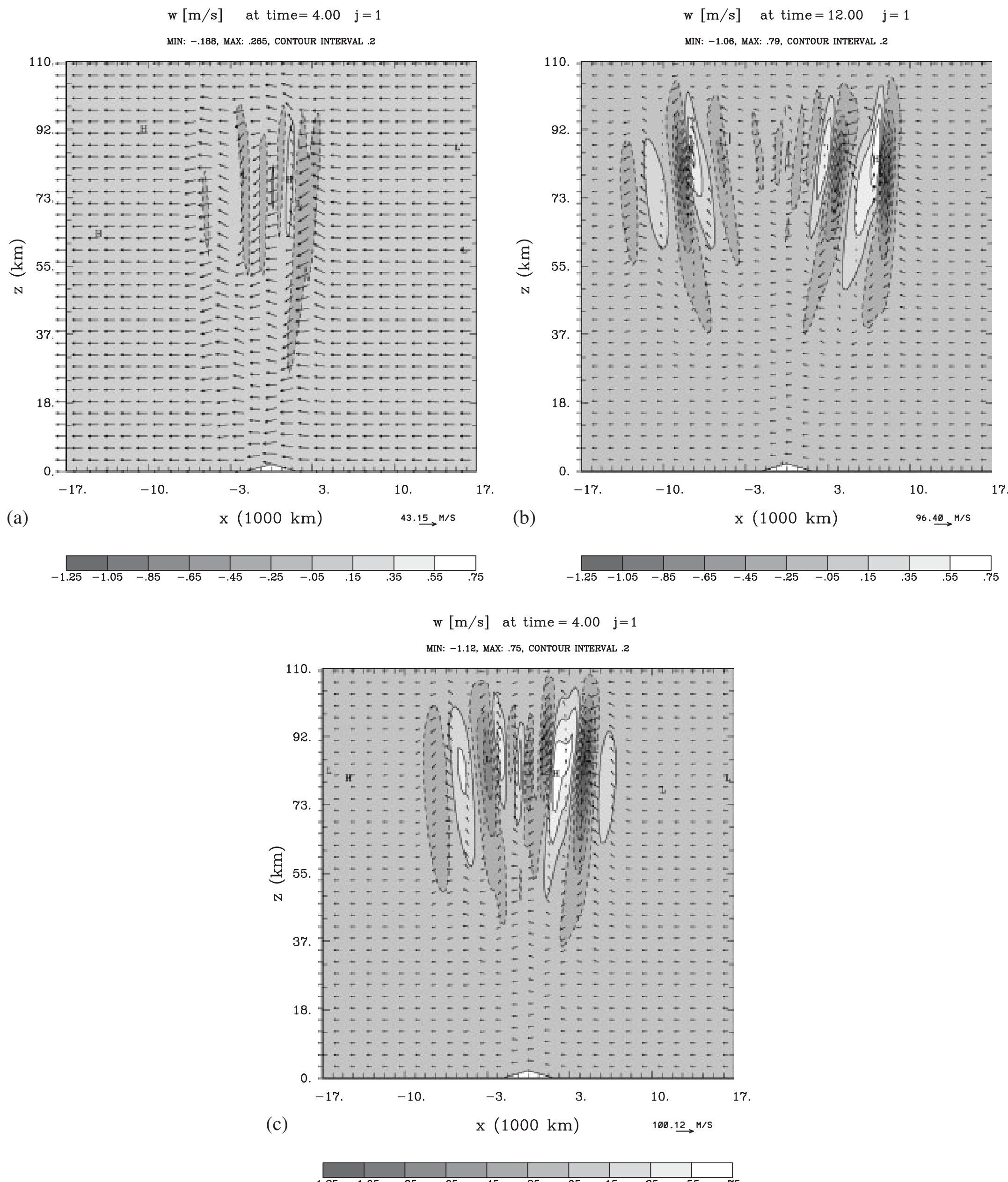

Figure 1. Vertical velocity for anelastic (a), (b) and Durran (c) solutions—idealized 2D deep planetary inertia-gravity wave at a mid latitude; (a) and (c) are after 4 h and (b) is after 12 h starting from a potential orographic flow. Gray scale and contour intervals are the same in all panels, but wind vectors scale with maximal flow magnitude. Mountain height and width allude to the continent of North America.

3. RESULTS

3.1. Inertia-gravity waves

For shallow (≈ 10 km deep) mesoscale motions the differences between the compressible, anelastic and Durran's systems were shown insignificant compared with truncation errors of discrete integrals [20]. In the numerical framework summarized in Section 2.2, we found the differences negligibly small even for deep mesoscale atmospheres and shallow planetary flows. For example, in benchmarks of a 60 km deep non-Boussinesq amplification of a 2D mountain wave [21] and a 8 km deep planetary 3D orographic flows [22], the differences between the anelastic and Durran's solutions were observed on third and second digit, respectively, in norms of the vertical-velocity fields. This corroborates the normal mode analysis in [10], revealing the differences between the two systems for deep internal gravity modes at longer horizontal scales. The latter is illustrated in Figure 1, which shows that long and deep inertia-gravity waves tend to propagate energy in the vertical too quickly in Durran's model.

3.2. Baroclinic life cycle experiments

We performed baroclinic life cycle experiments—relevant to the formation of synoptic-scale weather systems—with both sets of equations starting with identical initial conditions (a weakly perturbed baroclinic zonal jet, cf. Section 2 in [23]) in a 24 km deep domain. After 10 days of simulated time, the overall structure of the solution agrees for both equation systems. However, Durran's solution predicts a faster growth and slower propagation (by about 1 m/s) of baroclinic eddies, leading to substantial differences in solution details, Figure 2.

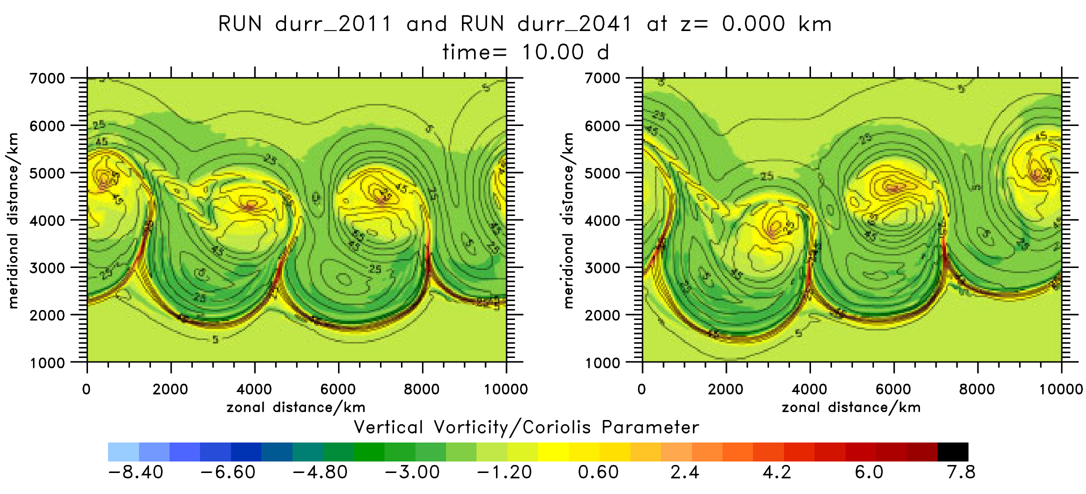

Figure 2. Vertical component of surface vorticity $(\nabla \times \mathbf{v} - \mathbf{f})/\|\mathbf{f}\|$ (color shaded) and horizontal wind speed (m/s) (solid contours) after 10 days of baroclinic life cycle experiments with the Lipps–Hemler (left) and Durran (right) equations.

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4. REMARK

For adiabatic dynamics the differences between consistent numerical solutions of the Durran and Lipps–Hemler anelastic equations appear immaterial for a broad range of problems, but become substantial for a deep planetary atmosphere. Whether this holds with diabatic forcing remains unknown since the resulting boundary value problem for pressure becomes more difficult to solve due to the additional heat source term from the energy equation.

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